Depth of powers

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The monomial ideals above are not square-free.....

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So $\mathrm{height}(\mathfrak{m}R(I)) \geq \min_k \{\mathrm{depth}(S/I^k)\} + 1$, with equality if $R(I)$ is Cohen-Macaulay. Now, let us remind that the **fiber cone** of I is the K-algebra:

$$F(I) = R(I)/\mathfrak{m}R(I).$$

(For instance, if I is generated by polynomials $f_1, \ldots, f_r$ of the *same* degree, then $F(I) = K[f_1, \ldots, f_r]$.)

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$$\begin{aligned}\dim(F(I)) &= \dim(R(I)) - \text{height}(\mathfrak{m}R(I)) \\ &\leq n + 1 - \min_k\{\text{depth}(S/I^k)\} - 1 \\ &= n - \min_k\{\text{depth}(S/I^k)\},\end{aligned}$$

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with equality if $R(I)$ is Cohen-Macaulay (these results are due to Burch and to Eisenbud-Huneke). So, it is evident that the study of depth-functions is closely related to the study of blow-up algebras.

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Theorem (Cowsik-Nori, 1976)

If I is radical, then:

$$\phi_I(k) = \dim(S/I) \ \forall \ k \iff I \text{ is a complete intersection}$$

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Theorem (_, Minh-Trung, 2011)

If $I = I_\Delta$ is a square-free monomial ideal, then:

$$\phi_I^s(k) = \dim(S/I)\ \forall\ k \iff \Delta \text{ is a matroid}$$

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If I is a summand ideal, then there exists a *minimal* system of generators $f_1, \ldots, f_r$ such that $K[f_1, \ldots, f_r]$ is a direct summand of S. In particular, if I is generated in a single degree, since all the minimal systems of generators of I generate the same K-algebra, one has to check the "summand" property for one given minimal system of generators.

Monomial ideals with constant depth-function

Given a monomial ideal $I \subseteq S = K[x_1, \ldots, x_n]$ minimally generated by monomials $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r}$, where $\mathbf{a}_i \in \mathbb{N}^n$, we denote by $\mathcal{M}(I)$ the monoid generated by $\mathbf{a}_1, \ldots, \mathbf{a}_r$.

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$$\mathcal{M}(I) = \mathrm{gp}(\mathcal{M}(I)) \cap \mathbb{N}^n.$$

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Lemma

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Example

$I = (ax^2, by^2, cxy) \subseteq K[a,b,c,x,y] = S$ is a degree-selection monomial ideal, but $\dim(R(I)) = 6 > 5 = \mathrm{depth}(R(I))$.

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The above facts lead to the following:

Questions

(i) Has any degree-selection ideal a constant depth-function?

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(i) $\operatorname{depth}(S/I^k) = 2 \ \forall\ 1 \leq k \leq 50$ ($\dim(S/I) = 3$);

(ii) If J is the polarization of I, suddenly $R(J)$ becomes CM.

The above facts lead to the following:

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Even if the above questions had a negative answer, it would nevertheless be interesting to find classes of monomial ideals satisfying the above hierarchies.

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Lemma (Zaimi)

For a monomial ideal $I \subseteq S$, the inclusion $K[\mathcal{M}(I)] \subseteq S$ is an algebra retract if and only if the minimal monomial generators of I are of the form $x_{\ell_1}u_1, \ \ldots, \ x_{\ell_r}u_r$ for some $\ell_1 < \ldots < \ell_r$ and monomials u_q coprime with $x_{\ell_1} \cdots x_{\ell_r}$ for any $q = 1, \ldots, r$.

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Let me notice that the above fact (disregarding the sentences in the parentheses) is true for *maximal* depth-functions (that is $\phi_I(k) = \dim(S/I)\ \forall\ k$).

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Let me notice that the above fact (disregarding the sentences in the parentheses) is true for *maximal* depth-functions (that is $\phi_I(k) = \dim(S/I)\ \forall\ k$). This is just because in this case I must be a monomial complete intersection, which has a CM Rees algebra and is easily seen to be a degree-selection.

Monomial ideals with constant depth-function

Another situation in which the previous question has an affirmative answer is when I is generated in degree 2 (i.e. $I = I(G)$ is an edge ideal), because the following characterization:

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- I is a degree-selection ideal;
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In this case, $R(I)$ is Cohen-Macaulay.

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