Average of polyhedral Morse inequalities & graded Betti numbers

Satoshi Murai (Osaka University)

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Notation

Δ : simplicial complex on $[n] = \{1, 2, \ldots, n\}$

$\mathbb{F}$: field

$$b_i(\Delta) = b_i(\Delta; \mathbb{F}) = \dim_{\mathbb{F}} H_i(\Delta; \mathbb{F})$$

$$\widetilde{b}_i(\Delta) = \widetilde{b}_i(\Delta; \mathbb{F}) = \dim_{\mathbb{F}} \widetilde{H}_i(\Delta; \mathbb{F})$$

$\Delta_W = \{F \in \Delta : F \subseteq W\}$: induced subcomplex

$\mathrm{lk}_\Delta(v) = \{F \in \Delta; F \cup \{v\} \in \Delta, v \notin F\}$: vertex link

Morse inequality for simplicial complex

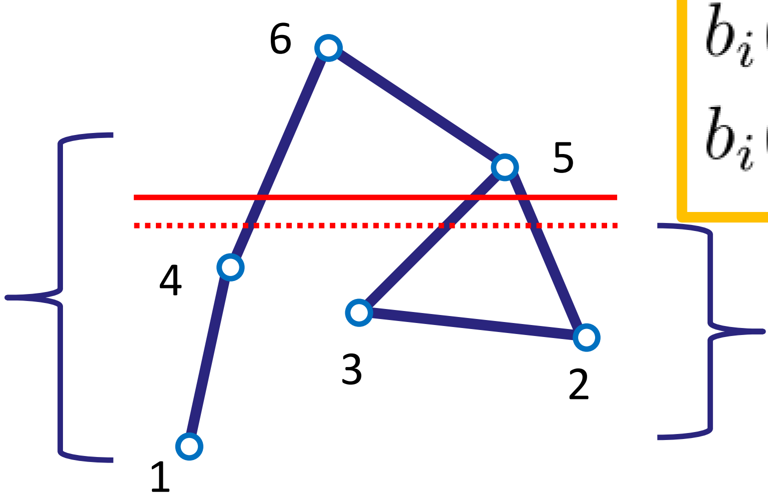
Morse Relation (Brehm–Kühnel '86)

If Δ *is a simplicial complex on* $[n]$, *then*

- $b_i(\Delta) \leq \sum_{k=1}^{n} \widetilde{b}_{i-1}(\mathrm{lk}_\Delta(k)_{[k-1]})$
- $\sum_{j=0}^{i}(-1)^{i-j} b_j(\Delta) \leq \sum_{j=0}^{i}(-1)^{i-j}\sum_{k=1}^{n}\widetilde{b}_{i-1}(\mathrm{lk}_\Delta(k)_{[k-1]})$

$$b_i(\Delta_{[k]}) \leq b_i(\Delta_{[k-1]}) + \tilde{b}_{i-1}(\mathrm{lk}_\Delta(k)_{[k]})$$

$\Delta_{[5]} \simeq$



$\simeq \Delta_{[4]}$

Average of Morse inequalities

Consider all permutatioins $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, n$.
Take the average of

$$b_i(\Delta) \leq \sum_{k=1}^{n} \widetilde{b}_{i-1}(\mathrm{lk}_\Delta(i_k)_{\{i_1,\ldots,i_{k-1}\}})$$

Corollary

$$b_i(\Delta) \leq \sum_{v:\text{vertex}} \left\{ \frac{1}{n} \sum_{W \subseteq [n] \setminus \{v\}} \frac{1}{\binom{n-1}{|W|}} \widetilde{b}_{i-1}(\mathrm{lk}_\Delta(v)_W) \right\}$$

(& same formula for alternating sums).

Average of Morse inequalities

Definition (Bagchi, Datta '14)

$$\widetilde{\sigma}_{i-1}(\Delta) = \frac{1}{n+1} \sum_{W \subseteq [n]} \frac{1}{\binom{n}{|W|}} \widetilde{b}_{i-1}(\Delta_W)$$

$$\mu_i(\Delta) = \sum_{v:\text{vertex}} \widetilde{\sigma}_{i-1}(\mathrm{lk}_\Delta(v)).$$

Corollary

- $b_i(\Delta) \leq \mu_i(\Delta)$
- $\sum_{j=0}^{i} (-1)^{i-j} b_j(\Delta) \leq \sum_{j=0}^{i} (-1)^{i-j} \mu_j(\Delta)$

Hochster's formula

$S = \mathbb{F}[x_1, \ldots, x_n]$: polynomial ring

$\mathbb{F}[\Delta] = S/I_\Delta$: Stanley-Reisner ring

$\beta_{i,j}(\mathbb{F}[\Delta]) = \dim_{\mathbb{F}} \mathrm{Tor}_i^S(\mathbb{F}[\Delta], \mathbb{F})_j$: graded Betti num

Theorem (Hochster's formula)

$$\beta_{i,j}(\mathbb{F}[\Delta]) = \sum_{|W|=j} \widetilde{b}_{i-1}(\Delta_W; \mathbb{F}).$$

Cor

$$\widetilde{\sigma}_{i-1}(\Delta) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{\binom{n}{k}} \beta_{k-i,k}(\mathbb{F}[\Delta]).$$

Research Targets

Δ: pure

- Δ is a **homology manifold** $\Leftrightarrow \mathrm{lk}_\Delta(v)$ is Gorenstein*
- Δ is a **combinatorial mfd** $\Leftrightarrow \mathrm{lk}_\Delta(v)$ is PL-sphere
- Δ is a triangulation of $X \Leftrightarrow ||\Delta|| \cong_{\mathrm{homeo}} X$

Combinatorial manifolds $\subset$ Triangulations of closed mfds $\subset$ Homology manifolds

Rmk: To apply $b_i(\Delta) \leq \mu_i(\Delta)$ for homology mfds, we need to study $\widetilde{\sigma}_{i-1}$ of Gorenstein* complex.

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First Application:

Minimal Triangulations of mfds

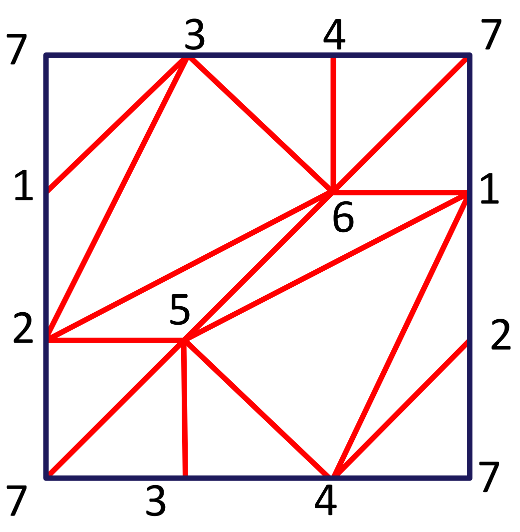
Minimal Triangulations

Question

For a given topological manifold M, how many vertices do we need to triangulate M?

<u>Example.</u>

7 vertices are required to triangulate $\mathbb{S}^1 \times \mathbb{S}^1$.



7 vert. triangulation of $\mathbb{S}^1 \times \mathbb{S}^1$

Kühnel's conjecture

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Conjecture (Kühnel)

If Δ is a combinatorial d-mfd with n vertices, then

$$\binom{n-d+j-2}{j+1} \geq \binom{d+2}{j+1} b_j(\Delta;\mathbb{Q}) \quad \left(j < \frac{d}{2}\right).$$

Conjecture I

If Δ is Gorenstein and has dim $d-1$, then*

$$\beta_{i,i+j}(I_\Delta) \leq \beta_{i,i+j}\big((x_1,\ldots,x_{n-d-1})^j\big) \quad \left(j < \frac{d}{2}+1\right).$$

Proposition

Conj. I $\Rightarrow$ Kühnel's Conj. (for homology mfds).

Conj. I ⇒ Kühnel's Conj.

Δ: homology manifold

$$b_j(\Delta) \le \mu_j(\Delta)$$

Morse inequality

$$= \sum_{v:\mathrm{vertex}} \widetilde{\sigma}_{j-1}(\mathrm{lk}_\Delta(v))$$

$$\le \frac{\binom{n-d-j-2}{j+1}}{\binom{d+2}{j+1}}$$

Substitute bounds in Conjecture I

Result

Conjecture I

If Δ is Gorenstein and has dim $d-1$, then*

$$\beta_{i,i+j}(I_\Delta) \leq \beta_{i,i+j}\big((x_1,\ldots,x_{n-d-1})^j\big) \quad \left(j < \frac{d}{2}+1\right).$$

Theorem (M)

- *Conjecture A holds for $j=2$.*
- *Conjecture A holds for simplicial polytopes.*

Remark

My second statement is an immediate consequence of

Theorem (Migliore–Nagel '03)

The Billera–Lee polytopes have the largest graded Betti numbers among all simplicial polytopes with the same f-vector.

Second Application:

Lower Bound Theorem

Notation

Δ: simplicial complex of dimension $d-1$

$h(\Delta) = (h_0(\Delta), h_1(\Delta), \dots, h_d(\Delta))$: h-vector

$$\left(\sum_{k \geq 0} \dim_{\mathbb{F}} (\mathbb{F}[\Delta])_k t^k = \frac{h_0(\Delta) + h_1(\Delta) t + \cdots + h_d(\Delta) t^d}{(1-t)^d} \right)$$

$$g_i(\Delta) = h_i(\Delta) - h_{i-1}(\Delta)$$

Lower Bound Theorem (LBT)

Barnette's Lower Bound Theroem (Barnette '73)

If Δ is the boundary complex of a simplicial polytope of dim ≥ 3, then $h_2(\Delta) \geq h_1(\Delta)$.

Theorem (Kalai '87, Fogelsanger '88)

If Δ is a normal pseudomanifold of dim ≥ 2, then $h_2(\Delta) \geq h_1(\Delta)$.

- pseudomanifold = pure, strongly connected, each codim 1 face is contained in exactly two facets.
- normal = link of codim >1 face is connected

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Strengthen of LBT

Theorem (Novik–Swartz '09 (conjectured by Kalai))

If Δ *is a homology manifold of dim* $d \geq 3$, *then*

$$h_2(\Delta) \geq h_1(\Delta) + \binom{d+2}{2} b_1(\Delta; \mathbb{Q}).$$

Theorem (M)

If Δ *is a normal pseudomanifold of dim* $d \geq 3$, *then*

$$h_2(\Delta) \geq h_1(\Delta) + \binom{d+2}{2} b_1(\Delta; \mathbb{F}).$$

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Idea of proof

Proposition

If Δ is a normal pseudomanifold of dim $d-1 \geq 2$, then

$$\beta_{i,i+2}(I_\Delta) \leq \beta_{i,i+2}\left((x_1, \ldots, x_{n-d-1})^2\right).$$

Proof of LBT for normal pseudomanifolds.

$$b_1(\Delta) - b_0(\Delta) \leq \mu_1(\Delta) - \mu_0(\Delta)$$ ← Morse Inequality

$$\leq \frac{1}{\binom{d+2}{2}} g_2(\Delta) - 1.$$ ← Substitute Bounds in the above Thm

Third Application:

Tight Triangulations

Tight Triangulation

Definition

Δ *is* **tight** $\Leftrightarrow b_i(\Delta) = \mu_i(\Delta)$ *for all* i.
($\Leftrightarrow \imath : H_i(\Delta) \to H_i(\Delta_W)$ *is surjective for all* i, W.)

Conjecture (Kühnel–Lutz '99)

Tight combinatorial manifold is vertex minimal.

Result

Conjecture (Kühnel–Lutz '99)

A combinatorial triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ $(i \leq j)$ *is tight if and only if it has exactly* $i + 2j + 4$ *vertices.*

Conjecture is true when $i = j$ (Kühnel)

Theorem (M)

Suppose $j > 2i$. *If a comb. triangulation of* $\mathbb{S}^i \times \mathbb{S}^j$ *is tight, then it has exactly* $i + 2j + 4$ *vertices.*

Idea of proof

Proposition

Δ: tight comb. triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ $(i < j)$. If $(I_{\mathrm{lk}_\Delta(v)})_{\leq i+1}$ has an $(i+1)$-linear resolution for each vertex v, then Δ has exactly $i+2j+4$ vertices.

$I_{\leq k}$: ideal generated by polynomials of deg $\leq k$ in I

Theorem (M)

Let Δ be a tight triang. of $\mathbb{S}^i \times \mathbb{S}^j$. If $j > 2i$, then $(I_{\mathrm{lk}_\Delta(v)})_{\leq i+1}$ has an $(i+1)$-linear resolution.

Idea of proof

Theorem (Herzog–Srinivasan)

If I is a monomial ideal generated in deg $\leq r$. Then

$$\max\{k \in \mathbb{Z} : \beta_{i+1,k}(I) \neq 0\}$$
$$\leq \max\{k \in \mathbb{Z} : \beta_{i,k}(I) \neq 0\} + r.$$

Corollary

If I is a monomial ideal generated in degree r and $\beta_{i,i+k}(I) = 0$ for all i and $k = r+1, \ldots, 2r-1$, then I has a linear resolution.

Conjectures

Conjecture I

Let Δ be a Gorenstein complex of dim $d-1$. Then* $\beta_{i,i+j}(I_\Delta) \leq \beta_{i,i+j}\left((x_1,\ldots,x_{n-d-1})^j\right) \quad (j < \frac{d}{2}+1)$.

Conjecture II

If Δ is a tight triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ $(i<j)$, then $(I_{\mathrm{lk}_\Delta(v)})_{\leq i+1}$ *has an* $(i+1)$*-linear resolution.*