Deviations of graded algebras

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Looking for more structure into complexes

When considering complexes of modules, elements appearing in different homological degrees "do not talk to each other": there is no prescribed way to multiply them together.

On the other hand, consider the Koszul complex on n elements:

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Here we **do** know how to multiply elements together! We are looking at an exterior algebra. Moreover, for all $x \in \bigwedge^i R^n$, $y \in \bigwedge^j R^n$,

$$\partial(xy) = \partial(x)y + (-1)^i x\partial(y).$$

Question

How can we generalize this?

Differential graded (DG) algebras

A *differential graded algebra* is a complex A endowed with an algebra structure $A \otimes A \to A$. By definition of tensor product of complexes, this implies that for all $a \in A_i$, $b \in A_j$ the so-called *skew Leibniz rule* must apply:

$$\partial(ab) = \partial(a)b + (-1)^i a\partial(b).$$

Following Avramov (*Infinite Free Resolutions*, our main source for this introduction), our differential graded algebras will also have two extra properties:

- $A_i = 0$ for all $i < 0$;
- *graded-commutativity*, i.e.

$$ab = (-1)^{ij} ba \text{ for all } a \in A_i, b \in A_j$$

and

$$c^2 = 0 \text{ for all } c \text{ of odd degree.}$$

Semi-free extensions

Given a DG-algebra A, we want to be able to extend it by adding new variables. Let z be a cycle of degree i.

- If i is even, we may add an *exterior* variable y of degree $i+1$ such that $\partial y = z$. This means that we are tensoring A by the exterior algebra on the free module generated by y. The differential of A is extended uniquely by requiring that the skew Leibniz rule holds.
- If i is odd, we can choose to add a *polynomial* or a *divided power* variable y of degree $i+1$ such that $\partial y = z$. This time we are tensoring A by the polynomial (or divided power) algebra on the free module generated by y. Again, we get a unique extension of the differential of A.

Extensions obtained by iterations of this procedure are called *semi-free extensions* of A.

Tate resolution

Let $R = S/I$ be a finitely generated (as an S-algebra) quotient of the Noetherian ring S. It is always possible to construct a DG-algebra resolution of R over S by running a process due to Tate.

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- Start by a single copy of S in degree 0, find a minimal set of generators $g_1, \dots, g_k$ for I and construct the DG-algebra extension $A^{(1)} = S\langle e_1, \dots, e_k \mid \partial e_i = g_i \rangle$ where all the e_i's have degree 1. This is the Koszul complex on $g_1, \dots, g_k$.

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- Find a minimal set of generators of $H_1(A^{(1)})$ and form the extension $A^{(2)}$ by adding to $A^{(1)}$ polynomial (or divided power) variables s_i of degree 2 such that each ∂s_i hits a generator. Note that $A^{(2)}$ has no homology in degree 1 by construction.

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- Find a minimal set of generators of $H_2(A^{(2)})$ and form $A^{(3)}$ by adding to $A^{(2)}$ exterior variables of degree 3 whose differentials hit the generators. $A^{(3)}$ has no homology in degree 1 nor 2 by construction.

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- And so on...

An example

$S = \mathbb{Q}[x,y],\ I = (x^2, xy),\ R = S/I.$

S

0

-

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$$
\begin{array}{ccc}
Se_1 & & \\
\bigoplus & \to & S \\
Se_2 & & \\
 & & \\
1 & & 0
\end{array}
$$

- $\partial e_1 := x^2,\ \partial e_2 := xy$

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$$\begin{array}{ccccc} & & Se_1 & & \\ Se_1 e_2 & \rightarrow & \oplus & \rightarrow & S \\ & & Se_2 & & \\ \\ 2 & & 1 & & 0 \end{array}$$

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$$\begin{array}{ccccc} Se_1e_2 & & Se_1 & & \\ \oplus & \to & \oplus & \to & S \\ {\color{red}Ss} & & Se_2 & & \\ & & & & \\ 2 & & 1 & & 0 \end{array}$$

- $\partial e_1 := x^2,\ \partial e_2 := xy$
- ${\color{red}\partial s := ye_1 - xe_2}$ $(\partial(ye_1 - xe_2) = y\partial(e_1) - x\partial(e_2) = yx^2 - x^2y = 0)$

An example

$S = \mathbb{Q}[x, y], \; I = (x^2, xy), \; R = S/I.$

$$\begin{array}{ccccccccccc} & & Sse_1e_2 & & Sse_1 & & Se_1e_2 & & Se_1 & & \\ \cdots & \to & \oplus & \to & \oplus & \to & \oplus & \to & \oplus & \to & S \\ & & Ss^2 & & Sse_2 & & Ss & & Se_2 & & \\ & & & & & & & & & & \\ & & 4 & & 3 & & 2 & & 1 & & 0 \end{array}$$

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$$\cdots \to \begin{matrix} Sse_1e_2 \\ \oplus \\ Ss^2 \end{matrix} \to \begin{matrix} Sse_1 \\ \oplus \\ Sse_2 \end{matrix} \to \begin{matrix} Se_1e_2 \\ \oplus \\ Ss \end{matrix} \to \begin{matrix} Se_1 \\ \oplus \\ Se_2 \end{matrix} \to S$$

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$$\begin{aligned} \partial(e_1e_2 + xs) &= \partial(e_1)e_2 - e_1\partial(e_2) + x\partial(s) \\ &= x^2e_2 - xye_1 + x(ye_1 - xe_2) = 0 \end{aligned}$$

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$S = \mathbb{Q}[x, y]$, $I = (x^2, xy)$, $R = S/I$.

$$
\begin{array}{ccccccccccc}
 & & & & Sse_1 & & & & & & \\
 & & Sse_1e_2 & & \oplus & & Se_1e_2 & & Se_1 & & \\
\dots & \to & \oplus & \to & Sse_2 & \to & \oplus & \to & \oplus & \to & S \\
 & & Ss^2 & & \oplus & & Ss & & Se_2 & & \\
 & & & & \color{red}{Sf} & & & & & & \\
 & & & & & & & & & & \\
 & & 4 & & 3 & & 2 & & 1 & & 0
\end{array}
$$

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$$
\begin{array}{ccccccccccc}
 & & Sse_1e_2 & & & & & & & & \\
 & & \oplus & & Sse_1 & & & & & & \\
 & & Ss^2 & & \oplus & & Se_1e_2 & & Se_1 & & \\
\dots & \to & \oplus & \to & Sse_2 & \to & \oplus & \to & \oplus & \to & S \\
 & & Sfe_1 & & \oplus & & Ss & & Se_2 & & \\
 & & \oplus & & Sf & & & & & & \\
 & & Sfe_2 & & & & & & & & \\
 & & 4 & & 3 & & 2 & & 1 & & 0
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$$

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Acyclic closures and minimal models

- If we perform the Tate process using only exterior and divided power variables, we are constructing an *acyclic closure* of R over S. We will denote it by $S\langle Y\rangle$, where $Y = \{Y_1, Y_2, Y_3, \dots\}$ is the set of added variables.
- If we use only exterior and polynomial variables, we speak of a *minimal model* of R over S. We will denote it by $S[X]$.

Attention!

Despite being the same in characteristic 0, acyclic closures and minimal models are quite different in general!

Theorem (Gulliksen 1968, Schoeller 1967)

Let $(R, \mathfrak{M}, \Bbbk)$ be a local ring. Then any acyclic closure of the residue field $\Bbbk$ over R is a minimal free resolution of $\Bbbk$ as an R-module.

Deviations

From now on let:

- $S = \Bbbk[T_1, \dots, T_n]$;
- I be a homogeneous ideal of S not containing any linear form;
- R be the quotient ring S/I.

(Note that Gulliksen-Schoeller's theorem holds in the graded case too.)

The Poincaré series of the residue field $\Bbbk$ as an R-module is the formal power series having the Betti numbers of $\Bbbk$ over R as coefficients. Let us write it in the following way:

$$\sum_{j=0}^{+\infty} \beta_j^R(\Bbbk) z^j =: P_{\Bbbk}^R(z) = \prod_{i=1}^{+\infty} \frac{(1+z^{2i-1})^{\varepsilon_{2i-1}}}{(1-z^{2i})^{\varepsilon_{2i}}}.$$

Definition

The number $\varepsilon_i := \varepsilon_i(R)$ is the i-th *deviation* of R.

Homological properties of deviations

As a consequence of Gulliksen and Schoeller's theorem, we find out how deviations fit in the previous picture:

Homological interpretation of deviations

The i-th deviation of R counts how many variables of degree i we have to introduce while constructing an acyclic closure of $\Bbbk$ as an R-module.

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Homological interpretation of deviations

The i-th deviation of R counts how many variables of degree i we have to introduce while constructing an acyclic closure of $\Bbbk$ as an R-module.

Theorem (Vanishing of deviations)

- $\varepsilon_1(R) = 0 \Leftrightarrow R$ *is a field.*
- $\varepsilon_2(R) = 0 \Leftrightarrow R$ *is regular.*
- *(Tate, Assmus)* $\varepsilon_3(R) = 0 \Leftrightarrow R$ *is a complete intersection* $\Leftrightarrow$ *all deviations from the third on vanish.*
- *(Halperin's rigidity theorem) If* $\varepsilon_j(R) = 0$ *for* ***some*** $j \geq 3$, *then* ***all*** *deviations from the third on vanish.*

Passing to the initial ideal

It is well known that Betti numbers of an ideal over a polynomial ring do not decrease when passing to the initial ideal (with respect to either a term order or a positive integral weight). What can be said about deviations?

Theorem (BDGMS 2014+)

Let $\omega \in \mathbb{N}^n$ be a positive integral weight. One has that, for all i,

$$\varepsilon_i(S/I) \leq \varepsilon_i(S/in_\omega(I)).$$

Since for any term order τ there exists a weight ω such that $in_\tau(I) = in_\omega(I)$, we have the same statement as above for term orders.

Lex ideals and deviations

In the same setting as before, let $L = Lex(I)$ be the lexicographic ideal of I, i.e. the ideal generated by the lexicographically first $HF(I, i)$ monomials of degree i for all i. A famous result states that

Theorem (Bigatti 1993, Hulett 1993, Pardue 1996)

$\beta_i^S(S/I) \leq \beta_i^S(S/L)$ for all i.

We prove that deviations have a similar behaviour:

Theorem (BDGMS 2014+)

$\varepsilon_i(S/I) \leq \varepsilon_i(S/L)$ for all i.

Since the Betti numbers of $\mathbb{k}$ over S/I are positive combinations of the deviations $\varepsilon_i(S/I)$, we recover the inequality due to Peeva (1996)

$$\beta_i^{S/I}(\mathbb{k}) \leq \beta_i^{S/L}(\mathbb{k}) \text{ for all } i.$$

Some asymptotics for special rational Poincaré series

Recall the formula

$$P_{\Bbbk}^{R}(z) = \prod_{i=1}^{+\infty} \frac{(1+z^{2i-1})^{\varepsilon_{2i-1}(R)}}{(1-z^{2i})^{\varepsilon_{2i}(R)}}.$$

What hypotheses can we require on the Poincaré series of $\Bbbk$ so that formal manipulations of this equality become feasible? A possible answer: assume that

$$P_{\Bbbk}^{R}(z) = \frac{(1+z)^c}{den(z)} = \frac{(1+z)^c}{\prod_{j=1}^{m}(1+\alpha_j z)},$$

where $den(0) = 1$ and the α_j's are nonzero complex numbers.
Notice that, although Poincaré series are not even rational in general, this is not an empty condition: both Golod and Koszul algebras satisfy this equality.

Some asymptotics for special rational Poincaré series / 2

Theorem (BDGMS 2014+)

If

$$P_{\Bbbk}^R(z) = \frac{(1+z)^c}{den(z)} = \frac{(1+z)^c}{\prod_{j=1}^m (1+\alpha_j z)}$$

(where $den(0) = 1$), then for all $i \geq 2$

$$\varepsilon_i(R) = \frac{(-1)^i}{i} \sum_{d|i} \mu\left(\frac{i}{d}\right) \sum_{j=1}^m \alpha_j^d,$$

where μ is the Möbius function.

This means that, in order to understand the asymptotic behaviour of the deviations of R, we "just" need to evaluate the roots of a certain polynomial!

A closer look at Golod and Koszul rings

By Pringsheim's theorem we know that the radius of convergence r of the Poincaré series is going to be a root (of smallest modulus) of the denominator $den(z)$. If R is not a complete intersection, then $0 < r < 1$ and hence the corresponding α is going to be greater than 1 in modulus.

- (BDGMS 2014+, essentially due to Sun and Avramov) If R is Golod, then $den(z)$ has a unique (and simple) root of minimum modulus. This implies that

  $$\varepsilon_i(R) \sim \frac{\rho^i}{i},$$

  where ρ is the reciprocal of r.
- (BDGMS 2014+) If R is Koszul, a result by Uliczka implies that $-r$ cannot be a root of $den(z)$. Hence, if $den(z)$ happens to have only real roots, deviations grow again exponentially (but this time we have to account for some multiplicative constant).

Is there only a root of minimum modulus?

In the Koszul case, $den(z)$ happens to be the h-polynomial of R (computed in $-z$). In general this object admits nonreal roots, but we are not aware of any example where two distinct roots of minimum modulus arise.

- (Chudnovsky and Seymour 2007) If G is a claw-free graph, its independence polynomial has only real roots. This implies that the h-polynomial of $S/I(G)$ (where $I(G)$ is the edge ideal of G) has only real roots.
- (Goldwurm and Santini 2000, Csikvári 2013) The independence polynomial of any graph G has a single root of minimum modulus. Note that one cannot infer directly from this that the h-polynomial of $S/I(G)$ has a single root of minimum modulus.

An interlude on Koszul homology

Given $R = S/I$, one can form the Koszul complex K^R, i.e. the Koszul complex on the variables $x_1, \dots, x_n$ in R. One can then consider its homology H^R.

Question

Which characteristics of R can be read off its Koszul homology H^R?

- R is a polynomial ring if and only if $H_i^R = 0$ for all $i > 0$.
- (Tate) R is a complete intersection if and only if H^R is the exterior algebra on H_1^R.
- (Avramov-Golod) R is Gorenstein if and only if H^R is a Poincaré algebra.

Connection to DG-algebra resolutions

How do DG-algebra resolutions come into play?
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On the other hand, since $S[X]$ is a DG-algebra resolution of R over S,

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We can hence study the Koszul homology of R by examining a minimal model for R over S.

Remark

Since $\dim_{\Bbbk} Tor_i^S(R, k)$ equals $\beta_i^S(R)$, studying the Betti table of R over S gives us information about the graded structure of the Koszul homology of R.

A fundamental theorem by Avramov

Why did we choose a minimal model of S over R and not, say, an acyclic closure? The following fundamental result gives a partial answer:

Theorem (Avramov)

Let $R\langle Y\rangle \twoheadrightarrow \Bbbk$ be an acyclic closure of $\Bbbk$ over R and let $S[X] \twoheadrightarrow R$ be a (not necessarily minimal) model of R over S. Then, for all $i \geq 1$,

$$\#X_i \geq \#Y_{i+1} = \varepsilon_{i+1}(R)$$

and equality holds if and only if the model is minimal.

A consequence of Avramov's theorem is that, if we have some precise knowledge of the deviations of R, we can exploit it while constructing a minimal model of R over S. This is actually how we constructed the example at the beginning of the talk!

Although it is very easy to write down the Koszul complex, understanding who the generators of its homology are turns out to be difficult.

Avramov posed the following question:

Question (Avramov)

If R is a Koszul algebra, is it true that H^R is generated by elements in its linear strand, i.e. elements of bidegree $(i, i+1)$?

The answer is no: Eisenbud and Caviglia found a counterexample using Macaulay2 and subsequently Conca and Iyengar, refining this example, were led to consider edge ideals of cycles of length $3k+1$ and inspect their Betti tables.

A special case: edge rings of cycles

Exploiting multigraded deviations, we constructed a partial minimal model for edge rings of cycles, obtaining the following corollary:

Theorem (BDGMS 2015)

Let $I(\mathcal{C}_n)$ denote the edge ideal of the n-cycle and let $R = S/I(\mathcal{C}_n)$. Then:

- *If $n = 3k$ or $n = 3k+2$, H^R is generated by $H^R_{1,2}$ and $H^R_{2,3}$.*
- *If $n = 3k+1$, H^R is generated by $H^R_{1,2}$, $H^R_{2,3}$ and by any nonzero $[z] \in H^R_{2k+1,3k+1}$.*

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Question

Does there exist a Koszul algebra R which is a domain and is such that its Koszul homology is not generated by its linear strand?

Thank you for your attention!